Article

A rigid triaxial rotator approximation of odd nuclei with quadrupole and octupole deformations

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Abstract: An approximation of a rigid asymmetric rotator for odd nuclei with quadrupole and octupole deformations have been developed and applied to describe energy levels of alternating-parity ground-state-band odd nuclei ^{161,163}Dy, ^{223,225}Th, ^{237}U and ^{239}Pu. The general structure of the sequence of low alternating-parity energy levels in odd nuclei in the lanthanide and actinide regions is reproduced, taking into account the K-mixing effect. Furthermore, interactions with an even-even nucleus and an unpaired nucleon are taken into account. In addition, the contribution of the unpaired nucleon to the collective motion of the entire system is taken into account by the Coriolis interaction.

Keyword: adiabatic collective model; quadrupole and octupole deformations; energy gap; effective triaxiality; ground-state-band; K-mixing effect.

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1. Introduction

The "energy gap" of the order of 1 to 2 MeV in the spectrum of a single-particle states of even-even nuclei simplifies the classification of collective excitations corresponding to the rotation of the nucleus as a whole and changes in surface shape. In odd nuclei, the energy of collective excitations differs little from the energy of single-particle excitations. The interaction of the rotating nucleus with the motion of the unpaired nucleon changes the structure of the rotational spectrum and complicates the classification of excited states of odd nuclei [1,2]. The low-lying excited states of odd nuclei are complex, and their rotational, vibrational, and single-particle excitation energies are indistinguishable. They correspond to specific values of parity and total angular momentum.

In even-even nuclei the even and odd angular momentum energy levels with positive and negative parities, respectively, due to the shape reflection asymmetry [3–6]. And in odd nuclei the structure of the energy spectrum is determined by the coupling between the reflection asymmetric even-even nucleus and the motion of the unpaired particle. Furthermore in heavy odd nuclei, the angular momentum of the ground state and its projection K are not unambiguously determined.

In the works [7–9], a collective model describing coherent quadrupole-octupole oscillations and rotations with Coriolis coupling between an even-even nucleus and an unpaired nucleon is applied to axially symmetric odd nuclei. However, in these studies, the contribution of transverse deformations and the K-mixing effect were not taken into account. In Ref. [7] Coriolis coupling is considered as the "soft" quadrupole-octupole oscillating core with the motion of the unpaired nucleon.

In the work [10], a collective rotation model with triaxial quadrupole and octupole deformations was considered under the adiabatic approximation. It was applied for the simultaneous description of the ground-state (g.s.) and the lowest negative-parity bands of the even–even nuclei $^{228-232}$Th, $^{230-238}$U and ^{240}Pu.

The purpose of this work is to quantitatively examine the possibility to explain the structure of alternating-parity bands within the triaxial quadrupole-octupole rotor concept based on the present experimental data [11] on positive- and negative-parity energy levels in heavy odd nuclei.

In Sec. 2, Bohr Hamiltonian for deformed odd nuclei is briefly recalled. Bohr Hamiltonian for effective triaxially deformed odd nuclei are presented in Sec. 3. The triaxial rigid rotator approximation for odd nuclei is given in Sec. 4. The results and concluding remarks are given in Sec. 5 and 6, respectively.

2. Bohr Hamiltonian for deformed odd nuclei

Bohr Hamiltonian for odd nuclei with quadrupole and octupole deformations with dynamical variables β_2 ($\beta_2 \geq 0$), β_3 ($\beta_3 \geq 0$), γ ($0^0 \leq \gamma \leq 60^0$), η ($0^0 \leq \eta \leq 90^0$), θ_1 ($0^0 \leq \theta_1 \leq 360^0$), θ_2 ($0^0 \leq \theta_2 \leq 180^0$), θ_3 ($0^0 \leq \theta_3 \leq 360^0$) has the following form:

$$\hat{H} = \hat{H}_{\beta_2} + \hat{H}_{\beta_3} + \hat{H}_\gamma + \hat{H}_\eta + \hat{H}_{\text{rot}} + \hat{H}_{\text{int}} + \hat{H}_{\text{p}}(x) + V(\beta_2, \beta_3, \gamma, \eta), \tag{1}$$

where

$$\hat{H}_{\beta_2} = -\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^4}\frac{\partial}{\partial\beta}\beta_2^4\frac{\partial}{\partial\beta_2}, \tag{2}$$

the operator of kinetic energy of β_2-vibrations, here B_2–quadrupole mass parameter.

$$\hat{H}_{\beta_3} = -\frac{\hbar^2}{2B_3}\frac{1}{\beta_3^4}\frac{\partial}{\partial\beta}\beta_3^4\frac{\partial}{\partial\beta_3}, \tag{3}$$

the operator of kinetic energy of β_3-vibrations, here B_3–octupole mass parameter.

$$\hat{H}_\gamma = -\frac{\hbar^2}{2B_2\beta_2^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}, \tag{4}$$

the operator of kinetic energy of γ-vibrations.

$$\hat{H}_\eta = -\frac{\hbar^2}{2B_3\beta_3^2}\left[\frac{\partial^2}{\partial\eta^2} + \frac{24\cos^2 2\eta - 6\cos 2\eta}{5 + 5\cos 2\eta + 8\cos^2 2\eta}\frac{\cos\eta}{\sin\eta}\frac{\partial}{\partial\eta}\right], \tag{5}$$

the operator of kinetic energy of η-vibrations [12,13].

$$\hat{H}_{\text{rot}} = \sum_{\kappa=1}^{3}\frac{\hbar^2(\hat{I}_\kappa - \hat{j}_\kappa)^2}{J_\kappa}, \tag{6}$$

the operator of rotational energy, here I_κ and $\hat{j}_\kappa$ (κ=1,2,3) are components off total angular momentum of the even-even nucleus (core) and an unpaired nucleon, respectively, J_κ-components of the moments of inertia tensor of the core and has the following form [14]:

$$J_1 = 8B_2\beta_2^2\sin^2\left(\gamma - \frac{2\pi}{3}\right) + 8B_3\beta_3^2\left[\frac{3}{2}\cos^2\eta + \sin^2\eta + \frac{\sqrt{15}}{2}\sin\eta\cos\eta\right],$$

$$J_2 = 8B_2\beta_2^2\sin^2\left(\gamma - \frac{4\pi}{3}\right) + 8B_3\beta_3^2\left[\frac{3}{2}\cos^2\eta + \sin^2\eta - \frac{\sqrt{15}}{2}\sin\eta\cos\eta\right],$$

$$J_3 = 8B_2\beta_2^2\sin^2\gamma + 8B_3\beta_3^2\sin^2\eta.$$

$\hat{H}_{\text{int}} + \hat{H}_{\text{p}}(x)$ Hamilton operator of the unpaired nucleon in the core field of the nucleus and $\hat{H}_{\text{p}}(x)$ takes into account the centrally symmetric part of the field, x – spatial and spin coordinates of an unpaired nucleon. Hamilton operator $\hat{H}_{\text{int}}$

$$\hat{H}_{\text{int}} = -2T_p\beta_2\left[\cos\gamma\left(3\hat{j}_3^2 - \hat{j}^2\right) + \sqrt{3}\sin\gamma\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right] -$$

$$-2T_p\beta_3\left[\cos\eta\left(3\hat{j}_3^2 - \hat{j}^2\right) + \sqrt{3}\sin\eta\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right], \tag{7}$$

takes into account the non-spherical part of the core field of the nucleus [2,15], T_p is the average kinetic energy of the unpaired particle in the nucleus [1].

The general solution of the Schrödinger equation with the operator (1) depends on the chosen potential energy $V(\beta_2, \beta_3, \gamma, \eta)$. Usually, this problem can be solved by choosing the phenomenological form of this potential energy. Often, the general form of potential energy is expressed in the following way [16]:

$$V(\beta_2, \beta_3, \gamma, \eta) = V(\beta_2) + V(\beta_3) + \frac{V(\gamma)}{\beta_2^2} + \frac{V(\eta)}{\beta_3^2}, \tag{8}$$

where V(β_2), V(β_3), V(γ) and V(η) are potential energies of β_2-, β_3-, γ- and η-vibrations, respectively.

A general solution of Schrödinger equation with the operator (1) has not yet been found. Therefore, the study of collective excitations of the deformed nuclei is carried out by introducing simplifying assumptions. One such assumption is the approximation of effective triaxially deformed odd nuclei.

3. Bohr Hamiltonian for effective triaxially deformed odd nuclei

Non-adiabatic approximation for triaxially deformed odd nuclei, where the variables γ, η are replaced by their effective values: γ_{eff}, η_{eff}. And variables β_3, β_2 are dynamic [2,14,15,17]. The Schrödinger equation with such a Hamilton operator

$$\left\{-\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^3}\frac{\partial}{\partial\beta_2}\left(\beta_2^3\frac{\partial}{\partial\beta_2}\right) - \frac{\hbar^2}{2B_3}\frac{1}{\beta_3^3}\frac{\partial}{\partial\beta_3}\left(\beta_3^3\frac{\partial}{\partial\beta_3}\right) + \right.$$

$$\left. + \hat{H}_{\text{rot}} + \hat{H}_{\text{int}} + \hat{H}_{\text{p}}(x) + V(\beta_2, \beta_3)\right\}\Psi^{\pm}_{Ij\tau}(x, \beta_2, \beta_3, \theta) = E_I\Psi^{\pm}_{Ij\tau}(x, \beta_2, \beta_3, \theta). \tag{9}$$

We will find the solution to this equation in the following form

$$\Psi^{\pm}_{Ij\tau}(x, \beta_2, \beta_3, \theta) = (\beta_2\beta_3)^{-\frac{3}{2}}\Phi^{\pm}_{Ij\tau}(x, \beta_2, \beta_3, \theta). \tag{10}$$

Then Schrödinger equation takes the form

$$\left\{\frac{1}{B_2}\frac{\partial^2}{\partial\beta_2^2} + \frac{1}{B_3}\frac{\partial^2}{\partial\beta_3^2} - \right.$$

$$\left. -\frac{2}{\hbar^2}\left[\hat{H}_{\text{rot}} + \hat{H}_{\text{int}}(p) + \hat{H}_{\text{p}}(x) + W(\beta_2, \beta_3) - E_I^{\pm}\right]\right\}\Phi^{\pm}_{Ij\tau}(x, \beta_2, \beta_3, \theta) = 0, \tag{11}$$

here

$$W(\beta_2, \beta_3) = V(\beta_2, \beta_3) + \frac{3}{B_2\beta_2^2} + \frac{3}{B_3\beta_3^2}.$$

We transit coordinates β_2 and β_2 in equation (11) to polar coordinates σ and ε [8]:

$$\beta_2 = \sqrt{\frac{B}{B_2}}\sigma\cos\varepsilon, \quad \beta_3 = \sqrt{\frac{B}{B_3}}\sigma\sin\varepsilon, \quad B = \frac{B_2 + B_3}{2}. \tag{12}$$

Schrödinger equation (11) for new variables

$$\left\{-\frac{\hbar^2}{2B}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma\partial\sigma} + \frac{\partial^2}{\sigma^2\partial\varepsilon^2}\right] + \hat{H}_{\text{rot}} + \hat{H}_{\text{int}}(p) + \hat{H}_{\text{p}}(x) + W(\sigma) - E_I^{\pm}\right\}\Phi_I^{\pm}(x, \sigma, \varepsilon, \theta) = 0. \tag{13}$$

The quadrupole-octupole oscillation wave function can be taken in a separable form

$$\Phi_I^{\pm}(\sigma, \varepsilon) = f_I^{\pm}(\sigma)\phi^{\pm}(x, \varepsilon, \theta), \tag{14}$$

where

$$\varphi^{\pm}_{Ij\tau}(x, \varepsilon, \theta) = \psi_{Ij\tau}(x, \theta)\phi^{\pm}(\varepsilon). \tag{15}$$

The equation for $\phi^{\pm}(\varepsilon)$ reads[7]:

$$\frac{\partial^2}{\partial\varepsilon^2}\phi^{\pm}(\varepsilon)+k^2\phi^{\pm}(\varepsilon)=0. \tag{16}$$

The solution of the Eq. (16) detailed considered in Ref. [7].

Wave functions $\psi_{Ij\tau}(x,\theta)$ satisfies following equation

$$\left(\hat{H}_{\text{rot}}+\hat{H}_{\text{int}}-\epsilon_{Ij\tau}\right)\psi_{Ij\tau}(x,\theta)=0. \tag{17}$$

Here $\epsilon_{I\tau}$ is the energy in relative units of energy of the triaxial rotor [10], while the quantum number $\tau = 1,2,3,$ ... labels the different eigenvalues $\epsilon_{Ij\tau}$ of $\hat{H}_{\text{rot}}+\hat{H}_{\text{int}}$ corresponding to given total angular momentum I and has following form:

$$\psi_{IMKj\Omega\tau}(x,\theta)=\frac{\sqrt{2I+1}}{4\pi}\sum_{K\geq 0}^{I}A_{Km}^{I\tau}\left[D_{MK}^{I}\chi_{\Omega}(x)\pm(-1)^{I+K}D_{M,-K}^{I}\chi_{-\Omega}(x)\right], \tag{18}$$

where $D_{MK}^{I}(\theta)$ is the Wigner function, $A_{Km}^{I\tau}$ – expansion coefficients, τ – labels the triaxial rigid rotator states. Here the quantum numbers M and K is the component of the angular momentum $\hat{I}$ on the third axes of the laboratory and intrinsic frames, respectively, Ω is the component of the angular momentum unpaired nucleon $\hat{j}$ to the third axis of the intrinsic frame, quantum number m=$(K-\Omega)/2$ is always integer [2]. $\chi_{\Omega}(x)$ is the eigenfunctions of the operator $\hat{H}_{\text{p}}(x)$ satisfying the equation:

$$\hat{H}_{\text{p}}(x)\chi_{\Omega}(x)=\mathcal{E}_j\chi_{\Omega}(x),$$

here $\mathcal{E}_j$ – eigenvalues of the operator $\hat{H}_{\text{p}}(x)$ [2,18].

The equation for $f_I^{\pm}(\sigma)$ reads:

$$-\frac{\hbar^2}{2B}\left[\frac{\partial^2(\sigma)}{\partial\sigma^2}+\frac{\partial}{\sigma\partial\sigma}\right]f_I^{\pm}+\left[\frac{\epsilon_{I\tau}+\mathcal{E}_j-\kappa^2}{\sigma^2}+V(\sigma)-E_I^{\pm}\right]f_I^{\pm}(\sigma)=0. \tag{19}$$

Note, that the components of the moment of inertia in polar coordinates has the form:

$$\mathfrak{J}_1=8B\sigma^2/a_1, \tag{20}$$

$$\mathfrak{J}_2=8B\sigma^2/a_2, \tag{21}$$

$$\mathfrak{J}_3=8B\sigma^2/a_3, \tag{22}$$

where

$$a_1=\left[\cos^2\varepsilon\sin^2\left(\gamma_{\text{eff}}-\frac{2\pi}{3}\right)+\sin^2\varepsilon\left(\frac{3}{2}\cos^2\eta_{\text{eff}}+\sin^2\eta_{\text{eff}}+\frac{\sqrt{15}}{2}\sin\eta_{\text{eff}}\cos\eta_{\text{eff}}\right)\right]^{-1}$$

$$a_2=\left[\cos^2\varepsilon\sin^2\left(\gamma_{\text{eff}}-\frac{4\pi}{3}\right)+\sin^2\varepsilon\left(\frac{3}{2}\cos^2\eta_{\text{eff}}+\sin^2\eta_{\text{eff}}-\frac{\sqrt{15}}{2}\sin\eta_{\text{eff}}\cos\eta_{\text{eff}}\right)\right]^{-1}$$

$$a_3=\left(\cos^2\varepsilon\sin^2\gamma_{\text{eff}}+\sin^2\varepsilon\sin^2\eta_{\text{eff}}\right)^{-1}.$$

In the next section, we consider the rigid rotator approximation for odd nuclei with quadrupole and octupole deformations [19]. We will implement this approximation in polar coordinates, as this version of the approximation under consideration is more useful [14,17] than the version with static parameters: $\beta_{2\text{eff}}$, γ_{eff}, $\beta_{3\text{eff}}$, η_{eff} [10].

4. The triaxial rigid rotator approximation for odd nuclei

To solve the equation (17) with the components of moment of inertia J_κ (20), (21), (22), we make the following assumption that the angular variable ε, can be replaced by its mean value $\varepsilon_0=\langle\varepsilon\rangle$, which in the adiabatic approximation can be considered as a general constant with respect to the rotational motion. We also replace the variable σ with its mean value $\sigma_0=\langle\sigma\rangle$.

As well as $\tilde{B} = 8B\sigma_0^2$ adjustable parameter. And energy levels will be expressed in the unit $\hbar^2/8B\sigma_0^2$.

Then, we re-write the rotational energy operator $\hat{H}_{\rm rot}$ (6) in the form [2]:

$$\hat{H}_{\rm rot} = \Gamma_1(\hat{I}^2 + \hat{j}^2 - \hat{I}_3^2 - \hat{j}_3^2) + \Gamma_2(\hat{I}_1^2 - \hat{I}_2^2 + \hat{j}_1^2 - \hat{j}_2^2) - 2\Gamma_2(\hat{I}_1\hat{j}_1 + \hat{I}_2\hat{j}_2) -$$

$$-2\Gamma_1(\hat{I}_1\hat{j}_1 - \hat{I}_2\hat{j}_2) + \Gamma_3(\hat{I}_3 - \hat{j}_3)^2 - (-1)^{I-j}a\left(I+\frac{1}{2}\right)\left(j+\frac{1}{2}\right)\delta_{K\frac{1}{2}}\delta_{m0},$$

here a–decoupling parameter with $K = 1/2$ [7].

$$\Gamma_1 = \frac{a_1 + a_2}{2}, \qquad \Gamma_2 = \frac{a_1 - a_2}{2}, \qquad \Gamma_3 = a_3.$$

And we will calculate matrix elements as [1] with taking into account K-mixing effect:

$$\langle IjK\Omega M|\hat{I}^2 + \hat{j}^2 - \hat{I}_3^2 - \hat{j}_3^2|I'j'K'\Omega'M'\rangle =$$

$$= I(I+1) + j(j+1) - K^2 - \Omega^2.$$

$$\langle IjK\Omega M|(\hat{I}_3 - \hat{j}_3)^2|I'j'K'\Omega'M'\rangle = (K-\Omega)^2.$$

$$\langle IjK\Omega M|\hat{I}_1\hat{j}_1|IjK\mp 1\Omega-1M\rangle = \pm\langle IjK\Omega M|\hat{I}_2\hat{j}_2|IjK\mp 1\Omega-1M\rangle =$$

$$= \frac{1}{4}\sqrt{(j+\Omega)(j-\Omega+1)(I\pm K)(I\mp K+1)}. \tag{23}$$

$$\langle IjK\Omega M|\hat{I}_1\hat{j}_1|IjK\mp 1\Omega+1M\rangle = \mp\langle IjK\Omega M|I_2 j_2|IjK\mp 1\Omega+1M\rangle =$$

$$= \frac{1}{4}\sqrt{(j-\Omega)(j+\Omega+1)(I\pm K)(I\mp K+1)}. \tag{24}$$

$$\langle K,\Omega|\hat{I}_1^2 - \hat{I}_2^2|\Omega, K\mp 2\rangle = \frac{1}{2}\sqrt{(I\pm K)(I\pm K-1)(I\mp K+1)(I\mp K+2)}$$

$$\hat{H}_{\rm int} = -\frac{1}{3\xi}\left\{\sin\varepsilon_0\left[\cos\gamma_{\rm eff}\left(3\hat{j}_3^2 - \hat{j}^2\right) + \sqrt{3}\sin\gamma_{\rm eff}\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right] + \right.$$

$$\left. + \cos\varepsilon_0\left[\cos\eta_{\rm eff}\left(3\hat{j}_3^2 - \hat{j}^2\right) + \sqrt{3}\sin\eta_{\rm eff}\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right]\right\}, \tag{25}$$

here $\xi = \hbar^2/6T_pB\sigma_0^2$ – coupling parameter of rotation with single-particle motion.

The unpaired nucleon contributes to the collective motion of the total system through the Coriolis interaction [20]. In present approximation, the contribution of the Coriolis interaction is taken into account by expressions (23) and (24). We would like to note that these expressions are more complex than similar expressions in [20], and this approximation is taken into account the contribution of the expression (25).

The coupling of the core and the unpaired nucleon in the core+particle system having the total wave function (18) provides a parity doublet structure of the spectrum [20]

$$I^\pm = I_0^\pm, (I_0+1)^\pm, (I_0+2)^\pm, (I_0+3)^\pm, ..., \tag{26}$$

where I_0 is the angular momentum (half integer) of the ground state.

Thus the intrinsic energy related to octupole degrees of freedom is separated in a part independent of the angular momentum, and an "alternating" part, which takes into account the shift down (for I^+) and the shift up (for I^-) of the intrinsic "octupole" energy with respect to E_0 (adjustable parameter of the theory).

$$E = \epsilon_{Ij\tau} + nE_0, \tag{27}$$

here n=0 for I^+ and n=1 for I^- [7]. It should be noted that the values of $\mathcal{E}_j$ and κ^2 in equation (18) remain constant for excited collective states, hence they can be eliminated.

5. Results

The following adjustable parameters are used in the triaxial rigid rotator approximation for odd nuclei: energy factor $\hbar\omega = \hbar^2/8B\sigma_0^2$ (in keV), ε_0 (in degree), γ_{eff} (in degree), η_{eff} (in degree), ξ (dimensionless), E_0 (in keV) and a (dimensionless).

The values of the adjusted model parameters $\hbar\omega$, γ_{eff}, η_{eff}, ε_0, E_0 ξ, a and the corresponding RMS deviations (in keV) obtained in this work for the alternating-parity spectra of several rare-earth and actinide odd nuclei: ^{161,163}Dy, ^{161,163}Dy, ^{223,225}Th, ^{237}U, ^{239}Pu are presented in the first column of the Table I.

The experimental data energy levels of the above-mentioned nuclei [11] are compared with the obtained theoretical results in the second and third columns of Table I. These comparisons show that the proposed rigid asymmetric rotator approximation provides a good description of the ground-state alternating-parity energy levels of the heavy odd nuclei in the lanthanide and actinide regions. Furthermore, the fourth column of this table presents results of the work [7]. Comparing the root-means-square values of our work and those of the work [7], the results of the latter appear preferable. However, it is necessary to take into account the difference between our approaches: the deformation variables are dynamic in [7], whereas in our approach, they are static. Note that the proposed model predicts the experimentally undetected alternating-parity energy levels of the odd nuclei under consideration.

Table 1. Energy levels of the odd nuclei.

Nucleus	I	Experiment	Theory	[7]
^{161}Dy	5/2$^+$	0	0	0
$\hbar\omega$ =2.03	7/2$^+$	43.8	21.7	55.6
γ_{eff}=22.5	9/2$^+$	100.4	96.9	126.9
η_{eff}=385.3	11/2$^+$	184.2	187.1	213.6
ε_0=20.4	13/2$^+$	267.4	292.7	315.4
E_0=31.6	15/2$^+$	407.0	413.9	432.2
ξ=0.11	17/2$^+$	508.1	550.7	563.5
RMS=32.16	19/2$^+$	718.6	702.3	709.0
RMS=39 [7]	21/2$^+$	826.1	869.4	868.4
	23/2$^+$	1118.3	1051.7	1041.2
	25/2$^+$	1221.8	1249.1	1227.2
	5/2$^-$	25.7	66.3	79.6
	7/2$^-$	103.1	129.5	135.0
	9/2$^-$	201.1	204.8	205.6
	11/2$^-$	320.7	294.8	292.2
	13/2$^-$	457.2	400.5	393.5
^{163}Dy	5/2$^+$	0	0	0
$\hbar\omega$=10.23	7/2$^+$	73.4	185.5	67.6
γ_{eff}=196.67	9/2$^+$	167.3	240.3	153.7
η_{eff}=80.21	11/2$^+$	281.6	323.1	258.7
ε_0=20.17	13/2$^+$	415.2	433.3	382.1
E_0=2.51	15/2$^+$	568.7	567.6	523.4
ξ=1.63	17/2$^+$	740.0	723.4	815.7
RMS=62.02	19/2$^+$	930.9	898.7	990.8
RMS=48 [7]	21/2$^+$	1137.1	1092.2	1182.6
	23/2$^+$	1363.9	1303.1	1390.7

Table 1. Continuation of the table 1.

Nucleus	I	Experiment	Theory	[7]
	$25/2^+$	1601.3	1530.8	1614.5
	$27/2^+$	1858.0	1775.4	1853.6
	$5/2^-$	250.9	135.3	137.9
	$7/2^-$	285.6	224.2	204.8
	$9/2^-$	336.5	278.9	290.5
	$11/2^-$	412.4	361.7	394.8
	$13/2^-$	497.2	471.9	517.3
	$15/2^-$	627.7	606.2	657.7
	$17/2^-$	734.3	762.1	682.5
	$19/2^-$	923.6	937.4	858.8
	$21/2^-$	1046.8	1130.9	1051.8
	$23/2^-$	1310.1	1341.7	1261.2
	$25/2^-$	1431.0	1569.5	1486.4
	$27/2^-$	1778.9	1814.1	1727.0
^{223}Th	$5/2^+$	0	0	0
$\hbar\omega$=4.97	$7/2^+$	51.3	80.8	53.8
γ_{eff}=10.95	$9/2^+$	118.9	144.8	122.5
η_{eff}=171.04	$11/2^+$	212.3	222.8	205.7
ε_0=10.48	$13/2^+$	320.0	314.7	303.0
E_0=3.48	$15/2^+$	428.7	420.5	448.6
ξ=5.29	$17/2^+$	569.6	539.9	572.0
RMS=26.89	$19/2^+$	706.0	672.8	707.9
RMS=14 [7]	$21/2^+$	858.1	819	855.9
	$23/2^+$	1021.6	978.2	1015.0
	$25/2^+$	1185.4	1150.1	1185.0
	$27/2^+$	1370.6	1334.4	1365.1
	$29/2^+$	1551.7	1530.9	1554.9
	$31/2^+$	1756.8	1739.1	1753.7
	$33/2^+$	1952.7	1958.8	1961.1
	$5/2^-$		56.4	36.5
	$7/2^-$		105.0	90.1
	$9/2^-$	180.5	168.6	158.5
	$11/2^-$	243.0	246.6	241.3
	$13/2^-$	324.1	338.5	338.2
	$15/2^-$	412.4	444.3	413.8
	$17/2^-$	547.3	563.7	537.7
	$19/2^-$	657.0	696.6	674.2
	$21/2^-$	838.1	842.7	822.6
	$23/2^-$	962.1	1001.9	982.4
	$25/2^-$	1179.4	1173.8	1152.9
	$27/2^-$	1313.8	1358.2	1333.7
	$29/2^-$	1558.4	1554.7	1524.1
	$31/2^-$	1702.5	1762.9	1723.6
	$33/2^-$		1935.9	1931.6
^{225}Th	$3/2^+$	0	0	0
$\hbar\omega$=4.59	$5/2^+$	31.0	40.1	41.8
γ_{eff}=15.97	$7/2^+$	68.0	87.3	78.7
η_{eff}=11.91	$9/2^+$	135.0	147.8	153.5
ε_0=47.15	$11/2^+$	187.0	221.4	211.0
E_0=1.97	$13/2^+$	303.0	308.1	318.0
ξ=0.63	$15/2^+$	370.0	407.5	395.6
RMS=20.5	$17/2^+$	530.0	519.6	533.8

Table 1. Continuation of the table 1.

Nucleus	I	Experiment	Theory	[7]
RMS=13 [7]	$19/2^+$	615.0	643.9	630.7
	$21/2^+$	807.0	780.5	799.0
	$23/2^+$	911.0	928.8	914.3
	$25/2^+$	1127.0	1088.8	1111.1
	$27/2^+$	1250.0	1260.2	1243.9
	$29/2^+$	1485.0	1443.1	1467.6
	$31/2^+$	1631.0	1637.1	1617.0
	$33/2^+$	1870.0	1842.4	1866.0
	$35/2^+$	2047.0	2058.9	2030.7
	$3/2^-$		14.6	13.4
	$5/2^-$		49.0	39.7
	$7/2^-$		97.0	98.1
	$9/2^-$		158.5	145.3
	$11/2^-$	254.0	230.2	236.3
	$13/2^-$	326.0	316.8	303.9
	$15/2^-$	433.0	416.3	426.6
	$17/2^-$	520.0	528.4	513.9
	$19/2^-$	668.0	652.7	667.3
	$21/2^-$	769.0	789.2	773.5
	$23/2^-$	957.0	937.6	956.1
	$25/2^-$	1072.0	1097.6	1080.3
	$27/2^-$	1291.0	1269.1	1290.6
	$29/2^-$	1426.0	1451.8	1431.8
	$31/2^-$	1658.0	1645.9	1668.3
	$33/2^-$	1824.0	1851.2	1825.4
	$33/2^-$	2057.0	2067.7	2086.3
^{237}U	$1/2^+$	0	0	0
$\hbar\omega$=5.63	$3/2^+$	11.4	49.1	20.3
γ_{eff}=48.1	$5/2^+$	56.3	86.6	51.5
η_{eff}=46.46	$7/2^+$	82.9	144.8	98.4
ε_0=136.31	$9/2^+$	162.3	204.9	154.0
E_0=14.35	$11/2^+$	204.1	280.3	226.8
ξ=4.51	$13/2^+$	317.3	375.2	305.8
RMS=41.91	$15/2^+$	375.1	455.4	403.1
RMS=30 [7]	$17/2^+$	518.2	581.0	504.2
	$19/2^+$	592.0	668.5	624.6
	$21/2^+$	762.8	816.4	746.2
	$23/2^+$	853.0	916.8	888.0
	$25/2^+$	1048.7	1082.7	1028.5
	$27/2^+$	1155.1	1198.9	1189.5
	$29/2^+$	1372.2	1381.2	1347.5
	$31/2^+$	1494.1	1514.1	1526.3
	$33/2^+$	1729.2	1712.2	1699.7
	$35/2^+$	1868.2	1861.9	1894.4
	$37/2^+$	2117.2	2075.6	2081.7
	$39/2^+$	2272.2	2242.2	2290.6
	$41/2^+$	2530.1	2471.4	2490.5
	$43/2^+$	2702.5	2654.8	2712.1
	$45/2^+$	2963.8	2899.7	2922.9
	$47/2^+$	3154.5	3099.7	3155.8
	$49/2^+$	3415.8	3360.2	3376.5
	$51/2^+$	3625.5	3576.9	3619.5

Table 1. Continuation of the table 1.

Nucleus	I	Experiment	Theory	[7]
	$53/2^+$	3886.8	3852.9	3848.9
	$55/2^+$	4115.0	4086.3	4100.7
	$57/2^+$	4377.0	4377.8	4338.0
	$59/2^+$		4674.5	4597.7
	$1/2^-$		330.9	375.2
	$3/2^-$		342.0	393.2
	$5/2^-$		372.4	425.4
	$7/2^-$		428.8	467.1
	$9/2^-$		486.6	524.6
	$11/2^-$		558.9	589.2
	$13/2^-$		650.4	671.0
	$15/2^-$	846.4	749.3	757.7
	$17/2^-$	930.0	874.8	862.4
	$19/2^-$	1027.5	962.3	969.8
	$21/2^-$	1131.0	1110.2	1096.1
	$23/2^-$	1250.7	1210.6	1222.8
	$25/2^-$	1376.1	1376.5	1369.1
	$27/2^-$	1515.7	1492.7	1513.4
	$29/2^-$	1662.3	1675.1	1678.0
	$31/2^-$	1821.8	1807.8	1838.5
	$33/2^-$	1987.7	2005.9	2019.8
	$35/2^-$	2166.5	2155.7	2194.9
	$37/2^-$	2349.7	2369.4	2391.2
	$39/2^-$	2547.5	2535.9	2579.5
	$41/2^-$	2746.7	2765.3	2789.3
	$43/2^-$	2960.5	2948.6	2989.4
	$45/2^-$	3174.7	3193.5	3211.2
	$47/2^-$	3401.5	3393.5	3422.0
	$49/2^-$	3630.0	3653.9	3654.6
	$51/2^-$	3865.0	3870.7	3874.9
	$53/2^-$	4105.0	4146.7	4117.1
	$55/2^-$	4344.0	4380.1	4345.8
	$57/2^-$	4597.0	4671.6	4596.7
	$59/2^-$	4835.0	4921.5	4832.9
^{239}Pu	$1/2^+$	0	0	0
$\hbar\omega$=3.334	$3/2^+$	7.9	57.2	12.6
γ_{eff}=45.417	$5/2^+$	57.3	80.9	55.3
η_{eff}=19.666	$7/2^+$	75.7	154.9	84.4
ε_0=7.283	$9/2^+$	163.8	177.1	160.7
E_0=12.425	$11/2^+$	193.5	285.3	206.0
ζ=0.876	$13/2^+$	318.5	324.2	314.8
RMS=76.75	$15/2^+$	359.2	450.5	375.6
RMS=63 [7]	$17/2^+$	519.5	518.9	515.4
	$19/2^+$	570.9	654.6	590.9
	$21/2^+$	764.7	756.7	760.0
	$23/2^+$	828.0	898.3	849.4
	$25/2^+$	1053.1	1032.4	1045.9
	$27/2^+$	1127.8	1180.9	1148.1
	$29/2^+$	1381.5	1342.9	1369.9
	$31/2^+$	1467.8	1501.7	1484.0
	$33/2^+$	1748.5	1687.2	1729.1
	$35/2^+$	1847.0	1859.5	1854.0

Table 1. Continuation of the table 1.

Nucleus	I	Experiment	Theory	[7]
	$37/2^+$	2152.2	2065.7	2120.3
	$39/2^+$	2263.0	2253.8	2255.2
	$41/2^+$	2590.1	2479.0	2540.7
	$43/2^+$	2714.0	2684.2	2684.6
	$45/2^+$	3060.1	2927.6	2987.6
	$47/2^+$	3198.0	3150.3	3139.5
	$49/2^+$	3559.1	3411.5	3458.3
	$51/2^+$	3713.0	3652.0	3617.5
	$53/2^+$	4087.1	3930.9	3950.5
	$55/2^+$	4256.0	4189.3	4116.2
	$1/2^-$	469.8	339.5	349.2
	$3/2^-$	492.1	352.7	3744.0
	$5/2^-$	505.6	376.4	394.2
	$7/2^-$	556.0	450.4	451.9
	$9/2^-$	583.0	472.6	488.0
	$11/2^-$	661.2	580.8	577.7
	$13/2^-$	698.7	619.6	629.1
	$15/2^-$	806.4	746.0	749.8
	$17/2^-$	857.5	814.3	816.0
	$19/2^-$	992.5	950.1	966.0
	$21/2^-$	1058.1	1052.1	1046.2
	$23/2^-$	1219.4	1193.7	1224.0
	$25/2^-$	1300.9	1327.9	1317.3
	$27/2^-$	1487.4	1476.4	1521.0
	$29/2^-$	1584.9	1638.3	1626.4
	$31/2^-$	1795.4	1797.2	1854.1
	$33/2^-$	1908.9	1982.6	1970.7
	$35/2^-$	2143.4	2155.0	2220.4
	$37/2^-$	2272.0	2361.2	2347.4
	$39/2^-$	2529.4	2549.3	2617.2
	$41/2^-$	2672.0	2774.5	2753.6
	$43/2^-$	2951.4	2979.7	3041.8
	$45/2^-$	3108.0	3223.1	3186.7
	$47/2^-$	3407.0	3445.8	3491.6
	$49/2^-$	3578.0	3707.0	3644.2
	$51/2^-$	3895.0	3947.5	3964.2
	$53/2^-$	4080.0	4226.3	4123.8
	$55/2^-$	4413.0	4484.8	4457.4

Conclusion

In this paper, we considered a triaxial quadrupole-octupole rotor to describe the structure of low energy level positive and negative parity in the spectra of heavy odd nuclei: ^{161,163}Dy, ^{223,225}Th, ^{237}U and ^{239}Pu. For ease of analysis, the deformation parameters in the components of the moments of inertia of the even-even core were converted to polar coordinates for reproduction of the general structure of low-energy sequences with alternating-parity in lanthanide and actinide odd nuclei, taking into account the K-mixing effect. Establishing the relationship between the various approaches to the problem of alternating-parity spectra in odd nuclei will require further, more detailed studies beyond the adiabatic approximation, including the corresponding vibrational modes.

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Conflict of interest The authors declare no conflict of interest.

References

[1] A. Bohr Dan. Mat. Fys. Medd. 26, no.14 (1952).
[2] A. S. Davydov *Excited states of atomic nuclei* [in Russian]. (Atomizdat, Moscow, 1967).
[3] H. J. Krappe and U. Wille, *Nucl. Phys. A* **124** (1969) 641.
[4] G. A. Leander, R. K. Sheline, P. Möller, P. Olanders, I. Ragnarsson, and A. J. Sierk, *Nucl. Phys. A* **388** (1982) 452.
[5] R. Jolos, P. von Brentano, and F. Dönau, *J. Phys. G* **19** (1993) L151.
[6] R. V. Jolos, and P. von Brentano, Phys. Rev. C 49, (1994). *Phys. Rev. C* **49** (1994) R2301.
[7] N. Minkov, S. Drenska, and P. Yotov, S. Lalkovski, D. Bonatsos, W. Scheid *Phys. Rev. C.* **76** (2007) 034324.
[8] A. Ya. Dzyublik and V. Yu. Denisov, *Yad. Fiz.* **56** (1993) 30. [Phys. At. Nucl. 56, 303 (1993)].
[9] V. Yu. Denisov and A. Ya. Dzyublik, *Nucl. Phys. A* **589** (1995) 17.
[10] M. S. Nadirbekov, N. Minkov, W. Scheid and M. Strecker, *Int. J. Mod. Phys. E* **25** (2016) 1650022.
[11] http://www.nndc.bnl.gov/ensdf/ Evaluated and Compiled Nuclear Structure Data: ENSDF and XUNDL Dataset Retrieval.
[12] M. S. Nadirbekov, S. N. Kudiratov, F. R. Kungurov *Acta Physica Polonica B* **56** (2025) 10-A2.
[13] M. S. Nadirbekov, S. N. Kudiratov *Journal of Fundamental and Applied Research* **5** (2025) 20250029.
[14] M. S. Nadirbekov, O. A. Bozarov, S. N. Kudiratov and N. Minkov *Phys. Scr.* **99** (2024) 095309.
[15] A. S. Davydov and R. A. Sardaryan, *Nucl. Phys.* **37** (1962) 106.
[16] F. Iachello, *Phys. Rev. Lett.* **87** (2001) 052502.
[17] M. S. Nadirbekov, O. A. Bozarov, S. N. Kudiratov, N. Minkov, *Int. J. Mod. Phys. E* **31** (2022) 2250078.
[18] S. G. Nilsson Dan. Mat. Fys. Medd. 29, no.16 (1955).
[19] J. Meyer-Ter-Vehn *Nuclear Physics A* **249** (1975) 111.
[20] N. Minkov, S. Drenska, M. Strecker and W. Scheid *J. Phys. G: Nucl. Part. Phys.* **36** (2009) 025108.